Holonomy algebras of pseudo-hyper-Kählerian manifolds of index 4

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Abstract

The holonomy algebra of a pseudo-hyper-Kählerian manifold of signature $(4, 4n + 4)$ is a subalgebra of $\mathfrak{sp}(1, n + 1)$. Possible holonomy algebras of these manifolds are classified. Using this, a new proof of the classification of simply connected pseudo-hyper-Kählerian symmetric spaces of index 4 is obtained.

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1. Introduction

The classification of holonomy algebras of Riemannian manifolds is well known and it has a lot of applications both in geometry and physics, see e.g. [5, 7, 11, 19, 21]. Lately the theory of pseudo-Riemannian geometries has been steadily developing. In particular, a classification of holonomy algebras of pseudo-Riemannian manifolds is an actual problem of differential geometry. It is solved only in some cases. The difficulty appears if the holonomy algebra preserves a degenerate subspace of the tangent space. Classification of holonomy algebras of Lorentzian manifolds is obtained in [25, 3, 24, 13, 15]; classification of holonomy algebras of pseudo-Kählerian manifolds of index 2 is achieved in [14]. These algebras are contained in $\mathfrak{so}(1, n + 1)$ and $\mathfrak{u}(1, n + 1) \subset \mathfrak{so}(2, 2n + 2)$, respectively. There are partial results for holonomy algebras of pseudo-Riemannian manifolds of signature $(2, n)$ and (n, n) [20, 4, 18]. More details can be found in the recent review [16].

In [10] holonomy algebras of pseudo-quaternionic-Kählerian manifolds with non-zero scalar curvature are classified. These algebras $\mathfrak{g}$ are contained in $\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)$ and they contain $\mathfrak{sp}(1)$. If $s \neq r$, then $\mathfrak{g}$ is irreducible. If $s = r$, then $\mathfrak{g}$ may preserve a degenerate subspace of the tangent space, in this case there are only two possibilities for $\mathfrak{g}$. This strong result follows mainly from the inclusion $\mathfrak{sp}(1) \subset \mathfrak{g}$.

Recall that a pseudo-hyper-Kählerian manifold is a pseudo-Riemannian manifold (M, g) together with three parallel g-orthogonal complex structures I_1, I_2, I_3 that satisfy the relations $I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}$, $I_3 = I_1 I_2 = -I_2 I_1$. Any such manifold has signature $(4r, 4s)$, $r + s > 1$, and its holonomy algebra $\mathfrak{g}$ is contained in $\mathfrak{sp}(r, s)$. Conversely, any simply connected pseudo-Riemannian manifold with such holonomy algebra is pseudo-hyper-Kählerian. Note that any pseudo-hyper-Kählerian manifold is also pseudo-quaternionic-Kählerian and it has zero scalar curvature.

In the present paper we classify all possible holonomy algebras $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ of pseudo-hyper-Kählerian manifolds of signature $(4, 4n + 4)$, $n \geq 1$. For $n = 0$ this classification is obtained in [9].

The Wu Theorem [27] allows to assume that the manifold (M, g) is locally indecomposable, i.e. locally it is not a product of pseudo-Riemannian manifolds of positive dimensions. This is the case if and only if the holonomy algebra $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ of (M, g) does not preserve any proper non-degenerate subspace of $\mathbb{R}^{4,4n+4}$ ($\mathbb{R}^{4,4n+4}$ is identified with the tangent space to the manifold (M, g) at some fixed point). Such subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ are called *weakly irreducible*. If the holonomy algebra $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ is irreducible, then

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$\mathfrak{g} = \mathfrak{sp}(1, n+1)$ [5, 11, 26]. Thus we may assume that $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ is weakly irreducible and not irreducible. We show that in this case $\mathfrak{g}$ preserves a 4-dimensional isotropic I_1, I_2, I_3-invariant subspace $W \subset \mathbb{R}^{4,4n+4}$. We identify $\mathbb{R}^{4,4n+4}$ with the pseudo-quaternionic Hermitian space $\mathbb{H}^{1,n+1}$ and denote by g the pseudo-quaternionic Hermitian metric on it. Then $W \subset \mathbb{H}^{1,n+1}$ is an isotropic quaternionic line. We fix a non-zero vector $p \in W$, then $W = \mathbb{H}p$. Let $q \in \mathbb{H}^{1,n+1}$ be any isotropic vector such that $g(p, q) = 1$. Denote by $\mathbb{H}^n$ the g-orthogonal complement to $\mathbb{H}p \oplus \mathbb{H}q$ in $\mathbb{H}^{1,n+1}$. Let $e_1, ..., e_n$ be a basis of $\mathbb{H}^n$ and let G be the corresponding Gram matrix of $g|_{\mathbb{H}^n}$, i.e. $G_{ab} = g(e_a, e_b)$. Denote by $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ the maximal subalgebra of $\mathfrak{sp}(1, n+1)$ that preserves the quaternionic isotropic line $W = \mathbb{H}p$, this Lie algebra has the matrix form:

$$\mathfrak{sp}(1, n+1)_{\mathbb{H}p} = \left\{ \mathrm{Op} \begin{pmatrix} a & -(G\bar{X})^t & b \\ 0 & \mathrm{Mat}_A & X \\ 0 & 0 & -\bar{a} \end{pmatrix} \middle| a \in \mathbb{H},\, A \in \mathfrak{sp}(n),\, X \in \mathbb{H}^n,\ b \in \mathrm{Im}\,\mathbb{H} \right\}. \tag{1}$$

Here $\mathrm{Op}\,\mathcal{M}$ denotes the $\mathbb{H}$-linear endomorphism of $\mathbb{H}^{1,n+1}$ given by a matrix $\mathcal{M}$, see Section 2. We get the decomposition

$$\mathfrak{sp}(1, n+1)_{\mathbb{H}p} = \mathbb{H} \oplus \mathfrak{sp}(n) \ltimes (\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H}). \tag{2}$$

We may also write $\mathbb{H} = \mathbb{R} \oplus \mathfrak{sp}(1)$, then

$$\mathfrak{sp}(1, n+1)_{\mathbb{H}p} = \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \ltimes (\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H}). \tag{3}$$

Let m, m_1, m_2 be integers such that either $m + m_1 + m_2 = n$ or $m + m_1 + m_2 \le n - 2$. Set the following denotation:

$$\begin{aligned} \mathbb{H}^m &= \mathrm{span}_{\mathbb{H}}\{e_1, ..., e_m\}, \\ \mathrm{Im}\,\mathbb{H}^{m_1} &= i\mathbb{R}^{m_1} \oplus j\mathbb{R}^{m_1} \oplus k\mathbb{R}^{m_1}, \quad \text{where} \quad \mathbb{R}^{m_1} = \mathrm{span}_{\mathbb{R}}\{e_{m+1}, ..., e_{m+m_1}\}, \\ \mathbb{C}^{m_2} &= \mathrm{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+m_1+1}, ..., e_{m+m_1+m_2}\}. \end{aligned}$$

Let L' be a real vector subspace of $\mathrm{span}_{\mathbb{H}}\{e_{m+m_1+m_2+1}, ..., e_n\}$ coinciding with a g-orthogonal direct sum of the real spaces of the form

$$\mathrm{span}_{\mathbb{R}}\{f_1, ..., f_l, if_1 + jf_2, ..., if_{l-1} + jf_l\}, \quad l \ge 2,$$

where we fix a fragmentation of the interval $[m + m_1 + m_2 + 1, ..., n]$ of natural numbers into a disjunct union of subintervals of length at least 2 and $f_1, ..., f_l$ are vectors from the set $\{e_{m+m_1+m_2+1}, ..., e_n\}$ corresponding to one of these subintervals.

Consider the following real vector subspace of $\mathbb{H}^n$:

$$L(m, m_1, m_2, L') = \mathbb{H}^m \oplus \mathrm{Im}\,\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2} \oplus L'. \tag{4}$$

Assume that the decomposition (4) is g-orthogonal. Let g be defined by this and the following conditions:

1) $g_{ab} = \delta_{ab}$, if $1 \le a, b \le m$;

2) $g_{ab} = \delta_{ab} + iw_{1ab} + jw_{2ab} + kw_{3ab}$, if $m + 1 \le a, b \le m + m_1$, where w_1, w_2, w_3 are skew-symmetric bilinear forms on $\mathbb{R}^{m_1}$;

3) $g_{ab} = \delta_{ab} + w_{ab}j$, if $m + m_1 + 1 \le a, b \le m + m_1 + m_2$, where w is a skew-symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^{m_2}$;

4) $g_{ab} = \eta_{ab} + i\Omega_{1ab} + j\Omega_{2ab} + k\Omega_{3ab}$, if $m + m_1 + m_2 + 1 \le a, b \le n$, where η_{ab} is a positive definite symmetric bilinear form on $\mathrm{span}_{\mathbb{R}}\{e_{m+m_1+m_2+1}, ..., e_n\}$ and $\Omega_1, \Omega_2, \Omega_3$ are skew-symmetric bilinear forms on $\mathrm{span}_{\mathbb{R}}\{e_{m+m_1+m_2+1}, ..., e_n\}$.

The above forms may be degenerate or zero.

Recall that any subalgebra $\mathfrak{h} \subset \mathfrak{sp}(n)$ can be decomposed as $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h})$, where $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ is the commutant of $\mathfrak{h}$ and $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$.

We prove the following theorem.

Theorem 1. *Let (M, g) be a locally indecomposable pseudo-hyper-Kählerian manifold of signature $(4, 4n + 4)$, $n \geq 1$. If the holonomy algebra $\mathfrak{g}$ of (M, g) is not irreducible, then $\mathfrak{g}$ is conjugated by an element of $\mathrm{SO}(4, 4n + 4)$ to one of the following subalgebras of $\mathfrak{sp}(1, n + 1)_{\mathbb{H}p}$:*

1) $\mathfrak{g}_1 = \mathbb{R} \oplus \mathfrak{h}_0 \oplus \mathfrak{h} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $0 \leq m \leq n$, $\mathfrak{h}_0 \subset \mathfrak{sp}(1)$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *are subalgebras,* $\mathfrak{h}_0 = \mathbb{R}i$ *or* $\mathfrak{h}_0 = \mathfrak{sp}(1)$. *If* $m < n$, *then* $\mathfrak{h}_0 = \mathbb{R}i$.

2) $\mathfrak{g}_2 = \mathbb{R} \oplus \{\phi(A) + A \,|\, A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $1 \leq m \leq n$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra,* $\phi : \mathfrak{h} \to \mathfrak{sp}(1)$ *is a non-zero homomorphism.*

If $m < n$, *then* $\operatorname{Im}\phi = \mathbb{R}i$, $\phi\,|_{\mathfrak{h}'} = 0$. *If* $m = n$, *then either* $\operatorname{Im}\phi = \mathbb{R}i$ *and* $\phi\,|_{\mathfrak{h}'} = 0$, *or* $\operatorname{Im}\phi = \mathfrak{sp}(1)$.

3) $\mathfrak{g}_3 = \mathfrak{h}_0 \oplus \{\varphi(A) + A \,|\, A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $0 \leq m \leq n$, $\mathfrak{h}_0 \subset \mathfrak{sp}(1)$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *are subalgebras,* $\varphi : \mathfrak{h} \to \mathbb{R}$ *is a linear map,* $\varphi\,|_{\mathfrak{h}'} = 0$.

If $m < n$, *then* $\mathfrak{h}_0 = \mathbb{R}i$ *and* $\varphi \neq 0$. *If* $m = n$, *then either* $\mathfrak{h}_0 = \mathbb{R}i$ *and* $\varphi \neq 0$, *or* $\mathfrak{h}_0 = \mathfrak{sp}(1)$.

4) $\mathfrak{g}_4 = \{\varphi(A) + \phi(A) + A \,|\, A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $0 \leq m \leq n$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra,* $\varphi : \mathfrak{h} \to \mathbb{R}$, $\phi : \mathfrak{h} \to \mathfrak{sp}(1)$ *are homomorphisms.*

If $m < n$, *then either* $\varphi = \phi = 0$ *or* $\varphi \neq 0$, $\operatorname{Im}\phi = \mathbb{R}i$ *and the maps* $i\varphi, \phi : \mathfrak{h} \to \mathbb{R}i$ *are not proportional,* $\varphi\,|_{\mathfrak{h}'} = \phi\,|_{\mathfrak{h}'} = 0$.

5) $\mathfrak{g}_5 = \mathbb{R}(\alpha + i) \oplus \{\varphi(A) + A \,|\, A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $0 \leq m \leq n$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra,* $\varphi : \mathfrak{h} \to \mathbb{R}$ *is a non-zero linear map with* $\varphi|_{\mathfrak{h}'} = 0$.

6) $\mathfrak{g}_6 = \mathfrak{h} \ltimes (L(m, m_1, m_2, L') \ltimes \operatorname{Im}\mathbb{H})$, *where* $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra.*

7) $\mathfrak{g}_7 = \{a - \mathrm{Op}(aE_{n-m}) | a \in \operatorname{Im}\mathbb{H}\} \oplus \mathfrak{h} \ltimes (\mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $n - m \geq 1$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra, and* $\mathrm{Op}(aE_{n-m}) \in \mathfrak{sp}(n - m)$ *is the element with the matrix* aE_{n-m}.

8) $\mathfrak{g}_8 = \{\phi(A) + A - \mathrm{Op}(\phi(A)E_{n-m}) | A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{n-m} \ltimes \operatorname{Im}\mathbb{H})$, *where* $n - m \geq 1$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra, and* $\phi : \mathfrak{h} \to \mathfrak{sp}(1)$ *is a surjective homomorphism*[1].

9) $\mathfrak{g}_9 = \{A + \psi(A) | A \in \mathfrak{h}\} \ltimes (\mathbb{H}^k \oplus V \ltimes \operatorname{Im}\mathbb{H})$, *here* $L(m, m_1, m_2, L') = \mathbb{H}^k \oplus V \oplus U$ *is an* η*-orthogonal decomposition* ($\eta = \operatorname{Re} g$), $\mathfrak{h} \subset \mathfrak{sp}(k)$ *is a subalgebra,* $\psi : \mathfrak{h} \to U$ *is a surjective linear map and* $\psi\,|_{\mathfrak{h}'} = 0$.

Conversely, all these algebras are Berger algebras.

To prove this theorem we use the fact that a holonomy algebra $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ is a Berger algebra, i.e. $\mathfrak{g}$ is spanned by the images of the algebraic curvature tensors $R \in \mathcal{R}(\mathfrak{g})$ of type $\mathfrak{g}$. Recall that $\mathcal{R}(\mathfrak{g})$ is the space of linear maps from $\wedge^2\mathbb{R}^{4,4n+4}$ to $\mathfrak{g}$ satisfying the first Bianchi identity. In [8] weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ containing the ideal $\mathcal{I} = \operatorname{Im}\mathbb{H}$ are partially classified. Here we find missing subalgebras, then we compute the spaces $\mathcal{R}(\mathfrak{g})$ for each of these algebras and we check which $\mathfrak{g}$ are Berger algebras. Then we show that each weakly-irreducible Berger subalgebra $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ contains $\mathcal{I}$. This gives the classification of weakly-irreducible not irreducible Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$. Remark that in the above theorem only possible holonomy algebras are listed. We do not know if all these algebras may appear as the holonomy algebras, to show this examples of manifolds must be constructed. Since the

[1] In the first version of this paper the Lie algebras $\mathfrak{g}_7$ and $\mathfrak{g}_8$ were missed, this is pointed out by Bastian Brandes. This mistake appeared since it was false stated that the intersection (26) below coincides with $\mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^{\perp_\eta})$ if $m_1 \neq 0$ or $L' \neq 0$.

most of the previosly known Berger algebras are realized as the holonomy algebras, one may expect that the algebras obtained in the above theorem can be realized as the holonomy algebras of pseudo-hyper-Kählerian manifolds.

In [1, 22, 23] simply connected pseudo-hyper-Kählerian symmetric spaces of index 4 are classified. In Section 4 we show that if the manifold (M,g) is locally symmetric, then $n=2$ and the holonomy algebra of (M,g) equals

$$L' \ltimes \operatorname{Im}\mathbb{H}, \quad \text{where} \quad L' = \operatorname{span}_{\mathbb{R}}\{e_1, e_2, je_1 + ie_2\}.$$

This together with the results of [9] gives a new proof of the classification of pseudo-hyper-Kählerian symmetric spaces of index 4. We give explicitly the curvature tensor of the obtained space.

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2. Preliminaries

Let $\mathbb{H}^m$ be an m-dimensional quaternionic vector space. A pseudo-quaternionic-Hermitian metric g on $\mathbb{H}^m$ is a non-degenerate $\mathbb{R}$-bilinear map $g : \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}$ such that $g(aX,Y) = ag(X,Y)$ and $\overline{g(Y,X)} = g(X,Y)$, where $a \in \mathbb{H}$, $X,Y \in \mathbb{H}^m$. Hence, $g(X,aY) = g(X,Y)\bar{a}$. There exists a basis $e_1,...,e_m$ of $\mathbb{H}^m$ and integers (r,s) with $r+s=m$ such that $g(e_t,e_l)=0$ if $t \neq l$, $g(e_t,e_t) = -1$ if $1 \leq t \leq p$ and $g(e_t,e_t) = 1$ if $p+1 \leq t \leq m$. The pair (r,s) is called the signature of g. In this situation we denote $\mathbb{H}^m$ by $\mathbb{H}^{r,s}$. The realification of $\mathbb{H}^m$ gives us the vector space $\mathbb{R}^{4m}$ with the quaternionic structure (i,j,k). Conversely, a quaternionic structure on $\mathbb{R}^{4m}$, i.e. a triple (I_1,I_2,I_3) of endomorphisms of $\mathbb{R}^{4m}$ such that $I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}$ and $I_3 = I_1I_2 = -I_2I_1$, allows us to consider $\mathbb{R}^{4m}$ as $\mathbb{H}^m$. A pseudo-quaternionic-Hermitian metric g on $\mathbb{H}^m$ of signature (r,s) defines on $\mathbb{R}^{4m}$ the i,j,k-invariant pseudo-Euclidian metric η of signature $(4r,4s)$, $\eta(X,Y) = \operatorname{Re}g(X,Y)$, $X,Y \in \mathbb{R}^{4m}$. Conversely, a I_1,I_2,I_3-invariant pseudo-Euclidian metric on $\mathbb{R}^{4m}$ defines a pseudo-quaternionic-Hermitian metric g on $\mathbb{H}^m$,

$$g(X,Y) = \eta(X,Y) + i\eta(X,I_1Y) + j\eta(X,I_2Y) + k\eta(X,I_3Y).$$

We will identify $(1,i,j,k)$ with (I_0,I_1,I_2,I_3), respectively. The identification $\mathbb{R}^{4r,4s} \simeq \mathbb{H}^{r,s}$ allows to multiply the vectors of $\mathbb{R}^{4r,4s}$ by quaternionic numbers.

The Lie algebra $\mathfrak{sp}(r,s)$ is defined as follows

$$\begin{aligned}\mathfrak{sp}(r,s) &= \{f \in \mathfrak{so}(4r,4s) | [f,I_1] = [f,I_2] = [f,I_3] = 0\} \\ &= \{f \in \operatorname{End}(\mathbb{H}^{r,s}) | \, g(fX,Y) + g(X,fY) = 0 \text{ for all } X,Y \in \mathbb{H}^{r,s}\}.\end{aligned}$$

Denote by $\mathfrak{sp}(1)$ the subalgebra in $\mathfrak{so}(4r,4s)$ generated by the $\mathbb{R}$-linear maps I_1,I_2,I_3.

Clearly, the tangent space of a pseudo-hyper-Kählerian manifold (M,g) at a point $x \in M$ one can identify with $(\mathbb{R}^{4r,4s}, \eta, I_1, I_2, I_3) = (\mathbb{H}^{r,s}, g)$. Then the holonomy algebra of a pseudo-hyper-Kählerian manifold is identified with a subalgebra $\mathfrak{g} \subset \mathfrak{sp}(r,s)$.

Let (V,η) be a pseudo-Euclidean space and $\mathfrak{g} \subset \mathfrak{so}(V)$ be a subalgebra. The space of curvature tensors $\mathcal{R}(\mathfrak{g})$ of type $\mathfrak{g}$ is defined as follows

$$\mathcal{R}(\mathfrak{g}) = \{R \in \operatorname{Hom}(\wedge^2 V, \mathfrak{g}) \mid R(u,v)w + R(v,w)u + R(w,u)v = 0 \textit{ for all } u,v,w \in V\}.$$

It is known that any $R \in \mathcal{R}(\mathfrak{g})$ satisfies

$$\eta(R(u,v)z,w) = \eta(R(z,w)u,v) \tag{5}$$

for all $u,v,w,z \in V$.

Denote by $L(\mathcal{R}(\mathfrak{g}))$ the vector subspace of $\mathfrak{g}$ spanned by the elements $R(u,v)$ for all $R \in \mathcal{R}(\mathfrak{g})$ and $u,v \in V$. *A subalgebra* $\mathfrak{g} \subset \mathfrak{so}(r,s)$ *is called a Berger algebra if* $L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}$. From the Ambrose-Singer

theorem it follows that if $\mathfrak{g} \subset \mathfrak{so}(V)$ is the holonomy algebra of a pseudo-Riemannian manifold, then $\mathfrak{g}$ is a Berger algebra. Therefore, Berger algebras may be considered as the candidates to the holonomy algebras.

Now we summarize some facts about quaternionic vector spaces. Let $\mathbb{H}^m$ be an m-dimensional quaternionic vector space and $e_1, ..., e_m$ a basis of $\mathbb{H}^m$. We identify an element $X \in \mathbb{H}^m$ with the column (X_t) of the left coordinates of X with respect to this basis, $X = \sum_{t=1}^m X_t e_t$. Let $f : \mathbb{H}^m \to \mathbb{H}^m$ be an $\mathbb{H}$-linear map. Define the matrix Mat_f of f by the relation $fe_l = \sum_{t=1}^m (\mathrm{Mat}_f)_{tl} e_t$. Now if $X \in \mathbb{H}^m$, then $fX = (X^t \,\mathrm{Mat}_f^t)^t$ and because of the non-commutativity of the quaternionic numbers this is not the same as $\mathrm{Mat}_f X$. Conversely, to an $m \times m$ matrix A of the quaternionic numbers we put in correspondence the linear map $\mathrm{Op}\, A : \mathbb{H}^m \to \mathbb{H}^m$ such that $\mathrm{Op}\, A \cdot X = (X^t A^t)^t$. If $f, g : \mathbb{H}^m \to \mathbb{H}^m$ are two $\mathbb{H}$-linear maps, then $\mathrm{Mat}_{fg} = (\mathrm{Mat}_g^t \,\mathrm{Mat}_f^t)^t$. Note that the multiplications by the imaginary quaternionic numbers are not $\mathbb{H}$-linear maps. Also, for $a, b \in \mathbb{H}$ holds $\overline{ab} = \bar{b}\bar{a}$. Consequently, for two square quaternionic matrices we have $(\overline{AB})^t = \bar{B}^t \bar{A}^t$.

Let $R \in \mathcal{R}(\mathfrak{sp}(r, s))$. Using (5) it is easy to show that for any $1 \leq \alpha \leq 3$ and $X, Y \in \mathbb{R}^{4r,4s}$,

$$R(I_\alpha X, Y) = -R(X, I_\alpha Y) \tag{6}$$

holds. Hence,

$$R(xX, Y) = R(X, \bar{x}Y) \tag{7}$$

for all $x \in \mathbb{H}$ and $X, Y \in \mathbb{R}^{4r,4s}$.

Let $W \subset \mathbb{R}^{4,4n+4}$ be an I_1, I_2, I_3-invariant isotropic subspace. Then W may be seen as an isotropic line in $\mathbb{H}^{1,n+1}$. Fix a nonzero vector $p \in W$ then $W = \mathbb{H}p$ and $g(p, p) = 0$. Let $q \in \mathbb{H}^{1,n+1}$ be any vector such that $g(q, q) = 0$ and $g(p, q) = 1$. Obviously, such vector exists and it is not unique. The restriction of g to the orthogonal complement E to $\mathbb{H}p \oplus \mathbb{H}q$ in $\mathbb{H}^{1,n+1}$ is positive definite, and we identify E with $\mathbb{H}^n$ and with $\mathbb{R}^{4n}$. Let $e_1, ..., e_n$ be a basis in $\mathbb{H}^n$.

Denote by $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ the maximal subalgebra of $\mathfrak{sp}(1, n+1)$ that preserves the quaternionic isotropic line $\mathbb{H}p$. This algebra has the matrix form (1). We denote the element from (1) by (a, A, X, b). One can easily find the following Lie brackets:

$$\begin{aligned} [(a,0,0,0),(a',0,X,b)] =& (a'a - aa', 0, \bar{a}X, 2\mathrm{Im} ba), \quad [(0,0,X,0),(0,0,Y,0)] = (0,0,0,2\mathrm{Im} g(X,Y)), \\ [(0,A,0,0),(0,B,X,0)] =& (0, [A,B]_{\mathfrak{sp}(n)}, AX, 0), \end{aligned}$$

where $a, a' \in \mathbb{H}$, $X, Y \in \mathbb{H}^n$, $A, B \in \mathfrak{sp}(n)$, $b \in \mathrm{Im}\mathbb{H}$. We get decomposition (3). The isomorphism $\{(a, 0, 0, 0) | a \in \mathrm{Im}\,\mathbb{H}\} \simeq \mathfrak{sp}(1)$ is given by $(a, 0, 0, 0) \mapsto -a$. The ideal $\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ is isomorphic to the quaternionic Heisenberg Lie algebra. The Levi-Malcev decomposition of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ has the form

$$\mathfrak{sp}(1, n+1)_{\mathbb{H}p} = \mathfrak{s} \ltimes \mathfrak{r}, \quad \mathfrak{s} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n), \quad \mathfrak{r} = \mathbb{R} \ltimes (\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H}),$$

where $\mathfrak{s}$ is a semisimple subalgebra and $\mathfrak{r}$ is the radical of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$. We may write the $\mathbb{Z}$-grading

$$\mathfrak{sp}(1, n+1)_{\mathbb{H}p} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \quad \mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n), \quad \mathfrak{g}_1 = \mathbb{H}^n, \quad \mathfrak{g}_2 = \mathrm{Im}\,\mathbb{H}$$

with the grading element $1 \in \mathbb{R} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$, i.e. $\mathrm{ad}_1|_{\mathfrak{g}_\alpha} = \alpha \mathrm{id}_{\mathfrak{g}_\alpha}$, $\alpha = 0, 1, 2$.

Coming back to $\mathbb{R}^{4,4n+4}$ we get the basis $p, I_1p, I_2p, I_3p, e_1, ..., I_3e_n, q, I_1q, I_2q, I_3q$ of $\mathbb{R}^{4,4n+4}$. To find the matrix form of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ acting in $\mathbb{R}^{4,4n+4}$ it is enough to change each element $c = c_0 + c_1 i + c_2 j + c_3 k$ of the matrix from (1) to the matrix

$$\begin{pmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & c_3 & -c_2 \\ c_2 & -c_3 & c_0 & c_1 \\ c_3 & c_2 & -c_1 & c_0 \end{pmatrix}.$$

3. Proof of Theorem 1

Since $\mathfrak{g}$ is weakly irreducible and not irreducible, $\mathfrak{g}$ preserves a degenerate vector subspace $V\subset\mathbb{R}^{4,4n+4}$. Let $V_1=V\cap V^\perp$, then V_1 is isotropic and $\dim V_1\leq 4$. Let $V_2=V_1^\perp\cap I_1V_1^\perp$. Clearly, $V_2\neq 0$ and it is degenerate, $\mathfrak{g}$-invariant and I_1-invariant. Then $V_3=V_2\cap V_2^\perp$ is isotropic, $\mathfrak{g}$-invariant and I_1-invariant. Starting with V_3 in the same way it can be shown that $\mathfrak{g}$ preserves an isotropic I_1,I_2-invariant subspace $W\subset\mathbb{R}^{4,4n+4}$, then W is also I_3-invariant and it has dimension 4. Consequently, $\mathfrak{g}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$.

The proof of the Theorem will consist of several parts.

3.1. The structure of the space $\mathcal{R}(\mathfrak{sp}(1,n+1)_{\mathbb{H}p})$

Let us find the space of curvature tensors $\mathcal{R}(\mathfrak{sp}(1,n+1)_{\mathbb{H}p})$ for the Lie algebra $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$.

Using the form η, the Lie algebra $\mathfrak{so}(4,4n+4)$ can be identified with the space

$$\wedge^2\mathbb{R}^{4,4n+4}=\mathrm{span}\{u\wedge v=u\otimes v-v\otimes u|u,v\in\mathbb{R}^{4,4n+4}\}$$

in such a way that $(u\wedge v)w=\eta(u,w)v-\eta(v,w)u$ for all $u,v,w\in\mathbb{R}^{4,4n+4}$. One can check that the element $\mathrm{Op}\begin{pmatrix} a & -(G\bar{X})^t & b\\ 0 & \mathrm{Mat}_A & X\\ 0 & 0 & -\bar{a}\end{pmatrix}\in\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ corresponds to the bivector

$$\begin{aligned}
&\Big(-a_0(p\wedge q+ip\wedge iq+jp\wedge jq+kp\wedge kq)+a_1(p\wedge iq-ip\wedge q+kp\wedge jq-jp\wedge kq)\\
&+a_2(-jp\wedge q+p\wedge jq-kp\wedge iq+ip\wedge kq)+a_3(-kp\wedge q+jp\wedge iq-ip\wedge jq+p\wedge kq)\Big)\\
&+A+\Big((p\wedge X_0+ip\wedge iX_0+jp\wedge jX_0+kp\wedge kX_0)+(p\wedge iX_1-ip\wedge X_1-jp\wedge kX_1+kp\wedge jX_1)\\
&+(p\wedge jX_2+ip\wedge kX_2-jp\wedge X_2-kp\wedge iX_2)+(p\wedge kX_3-ip\wedge jX_3+jp\wedge iX_3-kp\wedge X_3)\Big)\\
&+\Big(b_1(p\wedge ip-jp\wedge kp)+b_2(p\wedge jp+ip\wedge kp)+b_3(p\wedge kp-ip\wedge jp)\Big),
\end{aligned}$$

where $X=X_0+iX_1+jX_2+kX_3$, $X_0,...,X_3\in\mathbb{R}^n=\mathrm{span}_{\mathbb{R}}\{e_1,...,e_n\}\subset\mathbb{R}^{4n}\simeq\mathbb{H}^n$, $A\in\mathfrak{sp}(n)\subset\mathfrak{so}(4n)\simeq\wedge^2\mathbb{R}^{4n}$.

Let $\mathfrak{g}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$, $R\in\mathcal{R}(\mathfrak{g})$. The metric η defines the metric $\eta\wedge\eta$ on $\wedge^2\mathbb{R}^{4,4n+4}$. Using the above identification, R can be considered as the map

$R:\wedge^2\mathbb{R}^{4,4n+4}\to\mathfrak{g}\subset\mathfrak{so}(4,4n+4)\simeq\wedge^2\mathbb{R}^{4,4n+4}$. From (5), we obtain

$$\eta\wedge\eta(R(u\wedge v),z\wedge w)=\eta\wedge\eta(R(z\wedge w),u\wedge v)\tag{8}$$

for all $u,v,z,w\in\mathbb{R}^{4,4n+4}$. This shows that R is a symmetric linear map. Consequently R is zero on the orthogonal complement to $\mathfrak{g}$ in $\wedge^2\mathbb{R}^{4,4n+4}$. In particular, the vectors $q\wedge iq+jq\wedge kq$, $q\wedge jq-iq\wedge kq$, $q\wedge kq+iq\wedge jq$, $I_rp\wedge I_sp$, $I_rp\wedge X$, where $X\in\mathbb{R}^{4n}$, are contained in the orthogonal complement to $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$. Hence,

$$R(q,iq)=-R(jq,kq),\ R(q,jq)=R(iq,kq),\ R(q,kq)=-R(iq,jq),\ R(I_rp,I_sp)=R(I_rp,X)=0.\tag{9}$$

For a subalgebra $\mathfrak{h}\subset\mathfrak{so}(m)$ define the space

$$\mathcal{P}(\mathfrak{h})=\{P\in(\mathbb{R}^m)^*\otimes\mathfrak{h}|\eta(P(x)y,z)+\eta(P(y)z,x)+\eta(P(z)x,y)=0\text{ for all }x,y,z\in\mathbb{R}^m\},$$

where η is the scalar product on $\mathbb{R}^m$. This space is studied in [24, 17].

Proposition 1. *Any $R \in \mathcal{R}(\mathfrak{sp}(1,n+1)_{\mathbb{H}p})$ is uniquely defined by elements $C_{01}, C_{02} \in \mathbb{H}$, $A_{01}, A_{02}, A_{03} \in \mathfrak{sp}(n)$, $S_{01}, S_{02} \in \mathbb{H}^n$, $R' \in \mathcal{R}(\mathfrak{sp}(n))$, $P_0 \in \mathcal{P}(\mathfrak{sp}(n))$, $d_1, ..., d_5 \in \mathbb{R}$ in the following way:*

$$\begin{aligned} R(I_r p, I_s q) &= (0,0,0,B_{rs}), & R(I_s q, X) &= (0, P_s(X), T_s(X), \theta_s(X)),\\ R(X,Y) &= (0, R'(X,Y), L(X,Y), \tau(X,Y)), & R(I_r q, I_s q) &= (C_{rs}, A_{rs}, S_{rs}, D_{rs}),\\ R(I_r p, I_s p) &= R(I_r p, X) = 0, \qquad X, Y \in \mathbb{R}^{4n}, \end{aligned}$$

where

$$C_{03} = C_{02} i - C_{01} j, \quad C_{23} = -C_{01}, \quad C_{13} = C_{02}, \quad C_{12} = -C_{03}, \quad C_{rs} = C_{0r} I_s - C_{0s} I_r, \quad r,s \neq 0, \tag{10}$$

$$D_{01} = d_1 i + d_2 j + d_3 k, \quad D_{02} = d_2 i + d_4 j + d_5 k, \quad D_{03} = j D_{01} - i D_{02}, \tag{11}$$

$$D_{23} = -D_{01}, \quad D_{13} = D_{02}, \quad D_{12} = -D_{03}, \quad D_{rs} = I_r D_{0s} - I_s D_{0r}, \tag{12}$$

$$A_{23} = -A_{01}, \quad A_{13} = A_{02}, \quad A_{12} = -A_{03}, \tag{13}$$

$$T_0 = -\frac{1}{2}(I_1 A_{01} + I_2 A_{02} + I_3 A_{03}), \quad T_s = I_s T_0 - A_{0s} = -T_0 I_s, \quad s \neq 0, \tag{14}$$

$$P_s = -P_0 \circ I_s, \quad s \neq 0, \quad L(X,Y) = P_0(Y)X - P_0(X)Y, \tag{15}$$

$$B_{rs} = I_r C_{0s} + I_s B_{r0}, \quad B_{r0} = \frac{1}{2}(I_1 I_r C_{01} + I_2 I_r C_{02} + I_3 I_r C_{03}), \tag{16}$$

$$\tau(X,Y) = g(Y, T_0(X)) - g(X, T_0(Y)), \tag{17}$$

$$S_{03} = j S_{01} - i S_{02}, \quad S_{23} = -S_{01}, \quad S_{13} = S_{02}, \quad S_{12} = -S_{03}, \quad S_{rs} = I_r S_{0s} - I_s S_{0r}, \tag{18}$$

$$\theta_0(X) = \frac{1}{2}(I_1 g(X, S_{01}) + I_2 g(X, S_{02}) + I_3 g(X, S_{03})), \tag{19}$$

$$\theta_s(X) = g(X, S_{0s}) + I_s \theta_0(X) = -\theta_0(I_s X), \quad s \neq 0, \tag{20}$$

where $X, Y \in \mathbb{H}^n$. Moreover, it holds

$$\begin{aligned} \eta(L(Y,Z),X) = \eta(P_0(X)Y,Z), \quad & \eta(I_r \tau(X,Y)p, I_s q) = \eta(A_{rs}X, Y),\\ & \eta(I_r \theta_s(x)p, I_t q) = \eta(I_s S_{rt}, X), \quad \eta(I_t B_{rs} p, I_{t_1} q) = \eta(I_r C_{tt_1} p, I_s q), \end{aligned} \tag{21}$$

where $X, Y, Z \in \mathbb{R}^{4n}$.

Proof. Let $R \in \mathcal{R}(\mathfrak{sp}(1,n+1)_{\mathbb{H}p})$. The equality (9) shows that $R(I_r p, I_s p) = R(I_r p, X) = 0$. We may write

$$\begin{aligned} R(I_r p, I_s q) &= (\lambda_{rs}, F_{rs}, X_{rs}, B_{rs}), & R(I_s q, X) &= (\mu_s(X), P_s(X), T_s(X), \theta_s(X)),\\ R(X,Y) &= (\sigma(X,Y), R'(X,Y), L(X,Y), \tau(X,Y)), & R(I_r q, I_s q) &= (C_{rs}, A_{rs}, S_{rs}, D_{rs}). \end{aligned}$$

Now we find the conditions that satisfy the obtained elements. By the Bianchi identity,

$$R(I_r p, I_s q)X + R(I_s q, X)I_r p + R(X, I_r p)I_s q = 0.$$

Using the equality $R(X, I_r p) = 0$ and taking the projection on $\mathbb{H}^n$, we get $F_{rs}X = 0$, i.e. $F_{rs} = 0$. Using (5), we obtain

$$\eta(R(X,Y)I_r p, I_s q) = \eta(R(I_r p, I_s q)X, Y) = 0,$$

hence $\sigma(X,Y) = 0$. As in [9] one can prove that $\lambda_{rs} = 0$ and the equalities for C_{rs}, D_{rs}, B_{rs}. Writing down the Bianchi identity for the vectors $I_r p, I_s q, I_t q$ and taking the projection on $\mathbb{H}^n$, we get

$$I_t X_{rs} = I_s X_{rt}.$$

Hence, $X_{rs} = I_s X_{r0}$. Substituting this back to the above equation, we get $I_t I_s X_{r0} = I_s I_t X_{r0}$. Taking $t = 1$, $s = 2$, we obtain $X_{r0} = 0$. This shows that $X_{rs} = 0$. From this and (5) it follows that $\mu_s = 0$.

Writing down the Bianchi identity for the vectors $I_r q, I_s q, I_t q$ and taking the projection on $\mathbb{H}^n$, we get

$$I_t S_{rs} + I_r S_{st} + I_s S_{tr} = 0.$$

Taking $t = 0$, we get

$$S_{rs} = I_r S_{0s} - I_s S_{0r}.$$

Substituting this to the initial equality and taking $t = 1, r = 2, s = 3$, we get

$$S_{03} = jS_{01} - iS_{02}.$$

Note that $R(X,Y)Z = R'(X,Y)Z - g(Z, L(X,Y))p$. The Bianchi identity written for the vectors X, Y, Z implies $R' \in \mathcal{R}(\mathfrak{sp}(n))$. Moreover,

$$g(Z, L(X,Y)) + g(X, L(Y,Z)) + g(Y, L(Z,X)) = 0. \tag{22}$$

Hence,

$$\eta(L(X,Y), Z) + \eta(L(Y,Z), X) + \eta(L(Z,X), Y) = 0. \tag{23}$$

From the Bianchi identity written for the vectors q, X, Y it follows that

$$P_0(X)Y + L(X,Y) - P_0(Y)X = 0.$$

This and (23) imply $P_0 \in \mathcal{P}(\mathfrak{h})$. Using (6), we get $R(I_s q, X) = -R(q, I_s X)$, hence

$$P_s(X) = -P_0(I_s X), \quad T_s(X) = -T_0(I_s X), \quad \theta_s(X) = -\theta_0(I_s X), \quad s \neq 0.$$

The Bianchi identity written for the vectors $I_r q, I_s q, X$ implies

$$A_{rs} X + I_r T_s(X) - I_s T_r(X) = 0. \tag{24}$$

Taking $r = 0$, we get $T_s(X) = I_s T_0(X) - A_{0s}(X)$. Substituting this to (24) and taking $r = 1$, $s = 2$, we obtain

$$T_0 = -\frac{1}{2}(I_1 A_{01} + I_2 A_{02} + I_3 A_{03}).$$

Writing down the Bianchi identity for the vectors $I_r q, I_s q, X$ and taking the projection on $\mathbb{H}p$, we get

$$-g(X, S_{rs}) + I_r \theta_s(X) - I_r \theta_r(X) = 0.$$

Using this it is easy to get (19) and (20). The Bianchi identity applied to X, Y, q implies the equality for $\tau(X,Y)$ from (17).

We have proved that any $R \in \mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$ satisfies the conditions of the proposition. Conversely, it can be checked that any element R satisfying the conditions of the proposition belongs to $\mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$. $\square$

Denote by $\mathfrak{sp}(1,1)_{\mathbb{H}p}$ the subalgebra of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ that annihilates $\mathbb{H}^n \subset \mathbb{H}^{1,n+1}$. The space $\mathcal{R}(\mathfrak{sp}(1,1)_{\mathbb{H}p})$ is found in [9]. Note that any R given by elements C_{rs}, B_{rs}, D_{rs} and such that all the rest elements are zero belongs to $\mathcal{R}(\mathfrak{sp}(1,1)_{\mathbb{H}p})$. In particular, we get

Lemma 1. *Any subalgebra* $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ *such that* $\dim_{\mathbb{R}} \mathrm{pr}_{\mathbb{H}} \mathfrak{g} = 1$ *is not a Berger algebra.* $\square$

3.2. The algebras listed in the statement of the theorem are Berger algebras

Here we prove that the algebras listed in the statement of the theorem are Berger algebras.

Let $\mathfrak{g} = \mathfrak{g}_1$. If $m = n$ and $\mathfrak{h}_0 = \mathfrak{sp}(1)$, then any $R \in \mathcal{R}(\mathfrak{g})$ is given as in Proposition 1 with $A_{01}, A_{02}, A_{03} \in \mathfrak{h}$. Since the elements $C_{01} \in \mathbb{H}$, $A_{01} \in \mathfrak{h}$, $S_{01} \in \mathbb{H}^n$, $D_{01} \in \operatorname{Im}\mathbb{H}$ can be chosen in arbitrary way, $\mathfrak{g}$ is a Berger algebra. If $m = n$ and $\mathfrak{h}_0 = \mathbb{R}i$, then from [9] it follows that in addition to the above case $C_{01} = 0$ and $C_{02} \in \mathbb{R} \oplus \mathbb{R}i$ is arbitrary, hence $\mathfrak{g}$ is a Berger algebra. Suppose that $m < n$. Then $\mathfrak{h}_0 = \mathbb{R}i$. Each S_{rs} can be written as $S_{rs} = S'_{rs} + S''_{rs}$, where $S'_{rs} \in \mathbb{H}^m$ and $S''_{rs} \in \mathbb{C}^{n-m}$. Then $S''_{01}, S''_{02}, S''_{03} \in \mathbb{C}^{n-m}$. The condition $S''_{03} = jS''_{01} - iS''_{02}$ implies $S''_{01} = 0$, on the other hand, $S''_{02} \in \mathbb{C}^{n-m}$ is arbitrary. This shows that $\mathfrak{g}$ is a Berger algebra.

Let $\mathfrak{g} = \mathfrak{g}_2$. Suppose that $\operatorname{Im}\phi = \mathbb{R}i$. Let $R \in \mathcal{R}(\mathfrak{g})$ From the above example we get that $C_{01} = 0$. In addition, $C_{02} = c_1 + \phi(A_{02})$, where $c_1 \in \mathbb{R}$. Hence, $C_{03} = \phi(A_{02})i + c_1 i$. This shows that $c_1 = -i\phi(A_{03})$. Consequently $\mathfrak{g}$ is a Berger algebra. The case $\operatorname{Im}\phi = \operatorname{Im}\mathbb{H}$ will follow from this and the next case.

Let $\mathfrak{g} = \mathfrak{g}_4$ and $R \in \mathcal{R}(\mathfrak{g})$. Let $\phi = i\phi_1 + j\phi_2 + k\phi_3$, where the maps ϕ_1, ϕ_2, ϕ_3 take values in $\mathbb{R}$. Then

$$C_{rs} = \varphi(A_{rs}) + \phi_1(A_{rs})i + \phi_2(A_{rs})j + \phi_3(A_{rs})k.$$

The condition $C_{03} = C_{02}i - C_{01}j$ is equivalent to the equalities

$$\begin{aligned}
\phi_2(A_{01}) - \phi_1(A_{02}) = \varphi(A_{03}), &\quad \varphi(A_{02}) + \phi_3(A_{01}) = \phi_1(A_{03}),\\
\phi_3(A_{02}) - \varphi(A_{01}) = \phi_2(A_{03}), &\quad -\phi_2(A_{02}) - \phi_1(A_{01}) = \phi_3(A_{03}).
\end{aligned}$$

It is not hard to see that these conditions can be satisfied taking appropriate A_{01}, A_{02}, A_{03}. For example, if $\operatorname{Im}\phi = \mathbb{R}i$, then there exists a decomposition $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ such that $\ker\varphi = \mathfrak{h}_2 \oplus \mathfrak{h}_3$ and $\ker\phi = \mathfrak{h}_1 \oplus \mathfrak{h}_3$. In this case, it is enough to take $A_{01} \in \mathfrak{h}_3$, $A_{02} \in \mathfrak{h}_2$, $A_{03} \in \mathfrak{h}_1$ such that $\varphi(A_{03}) = -\phi_1(A_{02}) = 1$. This shows that $\mathfrak{g}$ is a Berger algebra.

The other Lie algebras from the statement of the theorem can be considered in the same way. For $\mathfrak{g}_8$ and $\mathfrak{g}_9$ note the following. Obviously, L' satisfies condition (28) given below. Let $X, Y, jX - iY \in L'$ and $S_{01} = X$, $S_{02} = Y$, then $S_{03} = jS_{01} - iS_{02} = jX - iY \in L'$, hence L' is spanned by S_{rs}.

3.3. Weakly-irreducible subalgebras of $\mathfrak{sp}(1, n+1)$ and real vector subspaces in $\mathbb{H}^n$

Now we review the classification of weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ obtained in [8] and make several corrections.

In [8] was constructed a homomorphism $f : \mathfrak{sp}(1, n+1)_{\mathbb{H}p} \to \mathfrak{sim}\mathbb{H}^n$, where $\mathfrak{sim}\mathbb{H}^n = \mathbb{R} \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \ltimes \mathbb{H}^n$ is the Lie algebra of the group $\operatorname{Sim}\mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. The homomorphism f is surjective with the kernel $\mathcal{I}$ and it is given as $f(a_0 + a_1, A, X, b) = (a_0, a_1 + A, X)$, where $a_0 \in \mathbb{R}$, $a_1 \in \mathfrak{sp}(1)$. Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly irreducible subalgebra and $L = \operatorname{pr}_{\mathbb{H}^n}\mathfrak{g} \subset \mathbb{H}^n$. It is shown that $\operatorname{span}_{\mathbb{H}} L = \mathbb{H}^n$ and the connected subgroup of $\operatorname{Sim}\mathbb{H}^n$ corresponding to $f(\mathfrak{g}) \subset \mathfrak{sim}\mathbb{H}^n$ preserves L and acts on it transitively. It was stated that there exists a g-orthogonal decomposition $L = L_1 \oplus L_2 \oplus L_3$ such that $L_1 \subset \mathbb{H}^n$ is a quaternionic subspace, $L_2 \subset \mathbb{H}^n$ is a real subspace such that $iL_2 = L_2$, $L_2 \cap jL_2 = 0$, $\operatorname{span}_{\mathbb{H}} L_2 = L_2 \oplus jL_2$, and $L_3 \subset \mathbb{H}^n$ is a real subspace such that $L \cap I_s L = 0$, $1 \le s \le 3$. Let $n = 1$ and $e \in \mathbb{H}$ be a non-zero vector, then the vector subspace $L = i\mathbb{R}e \oplus j\mathbb{R}e \oplus k\mathbb{R}e$ is missing. Next, it was stated that $\dim_{\mathbb{R}} L_3 = \dim_{\mathbb{H}} \operatorname{span}_{\mathbb{H}} L_3$. Let $n = 2$, e_1, e_2 a basis of $\mathbb{H}^2$, then the space $L = \operatorname{span}_{\mathbb{R}}\{e_1, e_2, je_1 + ie_2\}$ is missing. Finally it was stated that there exists a g-orthonormal basis of $\mathbb{H}^n$ consisting of vectors from L_1, L_2 and L_3, which is also not true in general.

Let $L \subset \mathbb{H}^n$ be a real subspace such that $\operatorname{span}_{\mathbb{H}} L = \mathbb{H}^n$. Put $L_1 = L \cap iL \cap jL \cap kL$, i.e. L_1 is the biggest quaternionic vector subspace in L. Let $L_2 = \{X \in L | g(X, L_1) = 0\}$, then $L = L_1 \oplus L_2$ and $L_2 \cap iL_2 \cap jL_2 \cap kL_2 = 0$. Note that the subspaces $L_2 \cap iL_2 \cap jL_2$, $L_2 \cap iL_2 \cap kL_2$, $L_2 \cap jL_2 \cap kL_2$, $iL_2 \cap jL_2 \cap kL_2$ can be taken to each other by i, j, k, and the intersection of any two of these subspaces are zero. Let L_3 be the direct sum of these subspaces. Let $L_4 = \{X \in L_2 | g(X, L_3) = 0\}$ and $L_5 = \{Y \in L_2 | g(Y, L_4) = 0\}$. Then $L_2 = L_5 \oplus L_4$ and

$$L_5 = L_3 \cap L_2 = iU \oplus jU \oplus kU = \mathfrak{sp}(1) \cdot U, \quad \mathfrak{sp}(1) = \operatorname{span}_{\mathbb{R}}\{i, j, k\}, \quad U = iL_2 \cap jL_2 \cap kL_2.$$

By the construction, it holds $I_rL_4\cap I_sL_4\cap I_tL_4=0$, if r,s,t are pairwise different. We see that L_5 is the biggest subspace of L_2 of the form $\mathfrak{sp}(1)\cdot V$, where $V\subset \mathrm{span}_{\mathbb{H}}L_2$ is a real subspace. In particular, this shows that the definition of L_5 does not depend on the choice of the generators I_1,I_2,I_3 of $\mathfrak{sp}(1)=\mathrm{span}_{\mathbb{R}}\{I_1,I_2,I_3\}$. Let $h_1\in\mathfrak{sp}(1)$ be an element with $h_1^2=-\mathrm{id}$ (any non-zero element of $\mathfrak{sp}(1)$ is proportional to such element). Let $L_4^1=h_1L_4\cap L_4$. Suppose that $L_4^1\neq 0$, then there is a g-orthogonal decomposition $L_4=L_4^1\oplus L_4'$ and it holds $h_1L_4'\cap L_4'$. Taking other $h_2\in\mathfrak{sp}(1)$, we may decompose L_4'. Clearly, this process is finite and we will get a g-orthogonal decomposition

$$L_4=L_4^1\oplus\cdots\oplus L_4^l\oplus L'$$

such that each L_4^α is h_α-invariant for some $h_\alpha\in\mathfrak{sp}(1)$ with $h_\alpha^2=-\mathrm{id}$ and $hL_4^\alpha\cap L_4^\alpha=0$ if $h\in\mathfrak{sp}(1)$ is not proportional to h_α. Next, $hL'\cap L'=0$ for any non-zero $h\in\mathfrak{sp}(1)$. Now we change the quaternionic structure $\mathfrak{sp}(1)=\mathrm{span}_{\mathbb{R}}\{I_1,I_2,I_3\}$ on $\mathbb{R}^{4n}$ to another quaternionic structure $\widetilde{\mathfrak{sp}(1)}=\mathrm{span}_{\mathbb{R}}\{\tilde I_1,\tilde I_2,\tilde I_3\}$ such that $\tilde I_1|_{\mathrm{span}_{\mathbb{H}}L_4^\alpha}=h_\alpha$. This means that we consider subalgebras of $\mathfrak{sp}(1,n+1)$ up to a conjugancy by elements of $\mathrm{SO}(4,4n+4)$. After such change we get

$$L=L_1\oplus L_5\oplus L_4^1\oplus L',$$

where L_1, L_5, L' satisfy the same properties as above and L_4^1 is I_1-invariant. Let $e_1,..,e_m$ be a g-orthonormal basis of $L_1\simeq\mathbb{H}^m$. Let $\mathbb{R}^{m_1}=iL_2\cap jL_2\cap kL_2$ and let $\{e_{m+1},...,e_{m+m_1}\}$ be an η-orthonormal basis of $\mathbb{R}^{m_1}$. Obviously, $L_5=i\mathbb{R}^{m_1}\oplus j\mathbb{R}^{m_1}\oplus k\mathbb{R}^{m_1}$. Let $X,Y\in\mathrm{span}_{\mathbb{H}}L_4'$. Note that the equality

$$h(X,Y)=\eta(X,Y)+i\eta(X,I_1Y)$$

defines a Hermitian metric on the complex space $\mathrm{span}_{\mathbb{H}}L_4'$. It holds

$$g(X,Y)=h(X,Y)+h(X,I_2Y)j.$$

Let $e_{m+m_1+1},...,e_{m+m_1+m_2}$ be an h-orthonormal basis of the complex space L_4'. Let $w(X,Y)=h(X,I_2Y)$, then the restriction of w to L_4' is a $\mathbb{C}$-linear skew-symmetric bilinear form.

To describe the structure of L' we use results from [12], where all real subspaces V of quaternionic vector spaces U are found. First a pair (V,U) of such spaces is called indecomposable if there are no pairs (V_1,U_1), (V_2,U_2) such that $U=U_1\oplus U_2$ and $V=V_1\oplus V_2$. In our case, L' may be decomposed into a g-orthogonal direct sum of real spaces V such that the pair (V,span_HV) is indecomposable. By our construction, it is enough to consider pairs (V,span_HV) such that $hV\cap V=0$ for any $h\in\mathfrak{sp}(1)$. Then we get only the following two possibilities:

$$V=A(2l-1)=\mathrm{span}_{\mathbb{R}}\{f_1,...,f_{l-1},f_{l+1},...,f_{2l-1},if_1+jf_2,...,if_{l-1}+jf_l,f_l+if_{l+1},jf_{l+1}+if_{l+2},...,jf_{2l-2}+if_{2l-1}\},$$

where $f_1,...,f_{2l-1}$ $(l\geq 2)$ is a basis of $\mathbb{H}^{2l-1}$, and

$$V=B(l)=\mathrm{span}_{\mathbb{R}}\{f_1,...,f_l,if_1+jf_2,...,if_{l-1}+jf_l\},$$

where $f_1,...,f_l$ $(l\geq 1)$ is a basis of $\mathbb{H}^l$.

We get that L is given by

$$L=\mathbb{H}^m\oplus\mathrm{Im}\,\mathbb{H}^{m_1}\oplus\mathbb{C}^{m_2}\oplus L',\tag{25}$$

i.e. as in Introduction, but at the moment L' is a g-orthogonal direct sum of vector spaces of the form $A(2l-1)$ and $B(l)$.

For the Lie algebra $f(\mathfrak{g})$ there are 3 possibilities:
$f(\mathfrak{g})=\mathfrak{f}_1=\mathbb{R}\oplus\bar{\mathfrak{h}}\ltimes L$, where $\bar{\mathfrak{h}}\subset\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)\cap\mathfrak{so}(L)\oplus\mathfrak{so}(L^\perp)$ is a subalgebra,
$f(\mathfrak{g})=\mathfrak{f}_2=\{\varphi(A)+A|A\in\bar{\mathfrak{h}}\}\ltimes L$, where $\bar{\mathfrak{h}}\subset\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)\cap\mathfrak{so}(L)\oplus\mathfrak{so}(L^\perp)$ is a subalgebra and $\varphi:\bar{\mathfrak{h}}\to\mathbb{R}$ is a linear map with $\varphi|_{\bar{\mathfrak{h}}'}=0$,
$f(\mathfrak{g})=\mathfrak{f}_3=\{A+\psi(A)|A\in\bar{\mathfrak{h}}\}\ltimes W$, where we have an η-orthogonal decomposition $L=W\oplus U$, $\bar{\mathfrak{h}}\subset\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)\cap\mathfrak{so}(L)\oplus\mathfrak{so}(L^\perp)$ is a subalgebra preserving W and annihilating U, $\psi:\bar{\mathfrak{h}}\to U$ is surjective linear map, $\psi|_{\bar{\mathfrak{h}}'}=0$.

Using this it is not hard to find all weakly irreducible subalgebras of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ containing the ideal $\mathcal{I}=\mathrm{Im}\,\mathbb{H}$. Namely if $f(\mathfrak{g})=\mathfrak{f}$ and $\mathfrak{g}$ contains $\mathrm{Im}\,\mathbb{H}$, then $\mathfrak{g}=\{(a_0+a_1,A,X,b)|(a_0,a_1+A,X)\in\mathfrak{f},\,b\in\mathrm{Im}\,\mathbb{H}\}$.

3.4. Classification of the Berger algebras containing $\mathrm{Im}\mathbb{H}$

Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly-irreducible subalgebra, let $\bar{\mathfrak{h}} = \mathrm{pr}_{\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)}\mathfrak{g}$ and $L = \mathrm{pr}_{\mathbb{H}^n}\mathfrak{g}$. Then $L \subset \mathbb{H}^n$ is a subspace as above and $\bar{\mathfrak{h}}$ preserves L. In particular, $\bar{\mathfrak{h}}$ is contained in the intersection

$$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^{\perp_\eta}). \tag{26}$$

Lemma 2. *Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly-irreducible Berger subalgebra. Then the following holds:*

1) *If $L' \neq 0$, then $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m)$, $\mathrm{pr}_{\mathbb{H}}\mathfrak{g} = 0$ and L' is a g-orthogonal sum of the spaces of type $B(l)$ with $l \geq 2$.*

2) *Suppose that $L' = 0$, then*

2.a) *if $m_1 \neq 0$ and $m_2 = 0$, i.e. $L = \mathbb{H}^m \oplus \mathrm{Im}\mathbb{H}^{m_1}$, $m + m_1 = n$, then $\mathrm{pr}_{\mathbb{R}}\mathfrak{g} = 0$, $\bar{h} \subset \{a + \mathrm{Op}(-aE_{m_1}) | a \in \mathrm{Im}\mathbb{H}\} \oplus \mathfrak{sp}(m)$, the projection of $\bar{\mathfrak{h}}$ to $\{a + \mathrm{Op}(-aE_{m_1}) | a \in \mathrm{Im}\mathbb{H}\}$ is either trivial or coincides with $\{a + \mathrm{Op}(-aE_{m_1}) | a \in \mathrm{Im}\mathbb{H}\}$;*

2.b) *if $m_1 = 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \mathbb{C}^{m_2}$, $m + m_2 = n$, then $\bar{h} \subset \mathbb{R}i \oplus \mathfrak{sp}(m)$;*

2.c) *if $m_1 \neq 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \mathrm{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2}$, $m + m_1 + m_2 = n$, then $\bar{h} \subset \mathfrak{sp}(m)$ and $\mathrm{pr}_{\mathbb{H}}\mathfrak{g} = 0$.*

Proof. 1) Suppose that $L' \neq 0$. Let $R \in \mathcal{R}(\mathfrak{g})$ be a tensor given as in Proposition 1. Then,

$$\mathrm{pr}_{\mathbb{H}\oplus\mathfrak{sp}(n)} R(q, I_s q) = C_{0s} + A_{0s} \in \mathbb{R} \oplus \bar{\mathfrak{h}}.$$

This shows that $C_{0s} + A_{0s}$ preserves L. It holds $A_{0s} = I_s T_0 + T_0 I_s$. Since T_0 takes values in L, for any $X \in L$ it holds

$$\mathrm{pr}_{\mathrm{span}_{\mathbb{H}} L'}(C_{0s}X + I_s T_0 X) \in L'. \tag{27}$$

Suppose that L' is of type $B(l)$, $l \geq 2$ and it is given by vectors $f_1, ..., f_l$. Then

$$\mathrm{pr}_{\mathrm{span}_{\mathbb{H}} L'} T_0 f_1 = \mathrm{pr}_{L'} T_0 f_1 = a_1 f_1 + \cdots + a_l f_l + b_1(i f_1 + j f_2) + \cdots + b_{l-1}(i f_{l-1} + j f_l),$$

where $a_1, ..., a_l, b_1, ..., b_{l-1} \in \mathbb{R}$. From (27) it follows that $C_{0s} f_1 + I_s \mathrm{pr}_{L'} T_0 f_1 \in L'$. Taking $s = 1$, we get

$$C_{01} f_1 + a_1 i f_1 + \cdots + a_l i f_l - b_1 f_1 \cdots - b_{l-1} f_{l-1} + k b_1 f_2 + \cdots + k b_{l-1} f_l \in L'.$$

Hence, $b_1 = \cdots = b_{l-1} = a_2 = \cdots = a_l = 0$ and $C_{01} = c_1 - a_1 i$ for some $c_1 \in \mathbb{R}$. In particular, $\mathrm{pr}_{L'} T_0 f_1 = a_1 f_1$. Similarly, we get $C_{02} = c_2 - a_1 j$ and $C_{03} = c_3 - a_1 k$ for some $c_2, c_3 \in \mathbb{R}$. Using the equality $C_{03} = C_{02} i - C_{01} j$, we get $a_1 = c_1 = c_2 = c_3 = 0$. Hence, $C_{rs} = 0$. This shows that $\bar{\mathfrak{h}} \subset \mathfrak{sp}(n)$ and $\mathrm{pr}_{\mathbb{H}}\mathfrak{g} = 0$. For L' of type $A(2l-1)$, $l \geq 2$, the proof is similar, hence $\bar{\mathfrak{h}} \subset \mathfrak{sp}(n)$ for any L'. Let $\mathfrak{h} = \bar{\mathfrak{h}}$. We claim that $\mathfrak{h}$ preserves decomposition (25). Since $\mathfrak{h}$ commutes with I_1, I_2, I_3, it preserves $\mathbb{H}^m = L \cap I_1 L \cap I_2 L \cap I_3 L$. Hence $\mathfrak{h}$ preserves $(\mathbb{H}^m)^{\perp_\eta} = L_2 = \mathrm{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2} \oplus L'$. Next, $\mathfrak{h}$ preserves $L_2 \cap j L_2 = i\mathbb{R}^{m_1} \oplus k\mathbb{R}^{m_1}$ and $L_2 \cap k L_2 = i\mathbb{R}^{m_1} \oplus j\mathbb{R}^{m_1}$, i.e. it preserves $i\mathbb{R}^{m_1}$. Thus $\mathfrak{h}$ preserves $\mathbb{R}^{m_1}$ and $\mathrm{Im}\mathbb{H}^{m_1}$. By similar arguments, $\mathfrak{h}$ preserves $\mathbb{C}^{m_2}$ and L'. The claim is proved.

The space $\mathbb{H}^n = \mathrm{span}_{\mathbb{H}} L$ is the direct sum of four subspaces and $\mathfrak{h}$ preserves this decomposition, hence we may write $A_{rs} = A^1_{rs} + A^2_{rs} + A^3_{rs} + A^4_{rs}$. Similarly, we decompose the elements S_{rs} and T_s. Equality (14) shows that $T^1_s = T_s|_{\mathbb{H}^m}$, $T^2_s = T_s|_{\mathbb{H}^{m_1}}$, $T^3_s = T_s|_{\mathbb{H}^{m_2}}$ and $T^4_s = T_s|_{\mathrm{span}_{\mathbb{H}} L'}$. Clearly, these maps take values in $\mathbb{H}^m$, $\mathrm{Im}\mathbb{H}^{m_1}$, $\mathbb{C}^{m_2}$ and L', respectively. Since $\mathfrak{h}$ preserves each summand in the direct sums $\mathbb{C}^{m_1} \oplus j\mathbb{C}^{m_1}$, $\mathbb{C}^{m_2} \oplus j\mathbb{C}^{m_2}$ and $L' \oplus iL'$, and acts in each summand simultaneously, according to [15], $R' \in \mathcal{R}(\mathfrak{h} \cap \mathfrak{sp}(m))$ and $P_0 \in \mathcal{R}(\mathfrak{h} \cap \mathfrak{sp}(m))$.

Let $S^2_{01} = is_1 + js_2 + ks_3$, $S^2_{02} = is_6 + js_4 + ks_5$, where $s_1, ..., s_6 \in \mathbb{R}^{m_1}$. The condition $S^2_{03} = jS^2_{01} - iS^2_{02} \in \mathrm{Im}\,\mathbb{H}^{m_1}$ is equivalent to the equality $s_6 = s_2$. The vectors $s_1, ..., s_5 \in \mathbb{R}^{m_1}$ are arbitrary.

Since $S^3_{01}, S^3_{02}, S^3_{03} \in \mathbb{C}^{m_2}$, and $S^3_{03} = jS^3_{01} - iS^3_{02}$, we see that $S^3_{01} = 0$ and $S^3_{02} \in \mathbb{C}^{m_2}$ may be arbitrary.

Since A_{rs} preserves $\mathrm{Im}\,\mathbb{H}^{m_1}$ and $A_{rs} \in \mathfrak{sp}(n)$, it preserves $i\,\mathrm{Im}\,\mathbb{H}^{m_1} \cap j\,\mathrm{Im}\,\mathbb{H}^{m_1} \cap k\,\mathrm{Im}\,\mathbb{H}^{m_1} = \mathbb{R}^{m_1}$. Let $X \in \mathbb{R}^{m_1}$, then

$$T^2_1(X) = -\frac{1}{2}(A^2_{01}(X) + I_3 A^2_{02}(X) - I_2 A^2_{03}(X)) \in \mathrm{Im}\,\mathbb{H}^{m_1},$$

hence, $A^2_{01}(X)=0$. Since $A^2_{01}\in\mathfrak{sp}(m_1)$, this implies $A^2_{01}=0$. Similarly, $A^2_{02}=A^2_{03}=0$.

Let $X\in\mathbb{C}^{m_2}$. Since

$$T^3_0(X)=-\frac{1}{2}(I_1A^3_{01}(X)+I_2A^3_{02}(X)+I_3A^3_{03}(X))\in\mathbb{C}^{m_2},$$

$I_2A^3_{02}(X)+I_3A^3_{03}(X)=I_2(A^3_{02}(X)-I_1A^3_{03}(X))$ and $A^3_{0r}(X)\in\mathbb{C}^{m_2}$ for any r, we get that $A^3_{02}(X)=I_1A^3_{03}(X)$. This implies $A^3_{02}|_{\mathbb{C}^{m_2}}=I_1A^3_{03}|_{\mathbb{C}^{m_2}}$. Hence, $A^3_{02}|_{\mathbb{C}^{m_2}}=A^3_{03}|_{\mathbb{C}^{m_2}}=0$ and $A^3_{02}=A^3_{03}=0$. Next, $T^3_2(X)=I_2T^3_0(X)-A^3_{02}(X)=\frac{1}{2}I_3A^3_{01}(X)$. Hence, $A^3_{01}(X)=0$. This shows that $A^3_{rs}=0$.

Let $Y\in L'$. Then for $s\neq 0$, $T^4_s(Y)=I_sT^4_0(Y)-A^4_{0s}(Y)\in L'$. Since $L'\cap I_sL'=0$ and A^4_{0s} preserves L', we get $T^4_0(Y)=0$. From (5) applied to the vectors I_sq, $X,Y\in\mathbb{H}^n$, q, it follows that

$$\eta(T_s(X),Y)=\eta(I_sT_0(Y),X).$$

Let $Y\in L'$, we get $\eta(T^4_s(X),Y)=0$ for any $X\in\mathbb{H}^n$ and any s. Hence T^4_s=0. Consequently, $A^4_{rs}=0$. Thus, $\mathfrak{h}\subset\mathfrak{sp}(m)$.

We see that L' must be spanned by elements $S_{01},S_{02},S_{03}\in L'$ that satisfy $S_{03}=jS_{01}-iS_{02}$, i.e. L' must satisfy

$$L'=\rho(L'),\quad\text{where}\quad\rho(L')=\mathrm{span}_{\mathbb{R}}\{X,Y,jX-iY|X,Y,jX-iY\in L'\}.\tag{28}$$

Clearly, the space $B(l)$, $l\geq 2$ satisfies this condition, while the space $B(1)$ does not satisfy this condition.

Lemma 3. *The space $L'=A(2l-1)$, $l\geq 2$ does not satisfy the condition* (28).

Proof. It can be directly checked that $\rho(A(3))=0$. We claim that if $L'=A(2l-1)$, $l\geq 3$, then

$$\rho(L')=\mathrm{span}_{\mathbb{R}}\{f_1,...,f_{l-1},f_{l+1},...,f_{2l-1},if_1+jf_2,...,if_{l-2}+jf_{l-1},jf_{l+1}+if_{l+2},...,jf_{2l-2}+if_{2l-1}\}.$$

We prove this claim using the induction over l. For $l=3$ this can be checked directly. Suppose that the claim holds for some $l\geq 3$. We will prove it for $l+1$. Clearly, $A(2(l+1)-1)$ can be obtained from $A(2l-1)$ adding some vectors f_0, f_{2l}, if_0+jf_1, $jf_{2l-1}+if_{2l}$. Let $X,Y\in A(2(l+1)-1)$. Then,

$$X=af_0+bf_{2l}+c(if_0+jf_1)+d(jf_{2l-1}+if_{2l})+\tilde{X},\quad Y=xf_0+yf_{2l}+u(if_0+jf_1)+v(jf_{2l-1}+if_{2l})+\tilde{Y}$$

for some $a,b,c,d,x,y,u,v\in\mathbb{R}$, $\tilde{X},\tilde{Y}\in A(2l-1)$. It can be checked that if $jX-iY\in A(2(l+1)-1)$, then $a,b,c,d=0$ and

$$jX-iY=j\tilde{X}_1-i\tilde{Y}-x(if_0+jf_1)-y(jf_{2l-1}+if_{2l})-uf_2-vf_{2l-2}+uf_0+vf_{2l},$$

where

$$\tilde{X}_1=\tilde{X}+xf_1-yf_{2l-1}-u(if_1+jf_2)-v(jf_{2l-2}+if_{2l-1})\in A(2l-1).$$

Hence, $j\tilde{X}_1-i\tilde{Y}\in A(2l-1)$. This and the induction hypothesis prove the inclusion $\subset$, the inverse inclusion is obvious. The lemma is proved. □

Thus L' is an g-orthogonal sum of the spaces of the form $B(l)$, $l\geq 2$.

2.a) Suppose that $L'=0$, $m_1\neq 0$ and $m_2=0$, i.e. $L=\mathbb{H}^m\oplus\mathrm{Im}\mathbb{H}^{m_1}$, $m+m_1=n$. For simplicity of the exposition we may assume that $m=0$, i.e. $L=\mathrm{Im}\mathbb{H}^n$. Obviously, $\bar{\mathfrak{h}}\cap\mathfrak{sp}(1)=0$ and elements of the form $a-\mathrm{Op}(aE_n)$ (where $a\in\mathrm{Im}\mathbb{H}$) preserve $L=\mathrm{Im}\mathbb{H}^n$, hence $\bar{\mathfrak{h}}\subset\{a-\mathrm{Op}(aE_n)|a\in\mathrm{Im}\mathbb{H}\}\oplus\tilde{\mathfrak{h}}$, where $\tilde{\mathfrak{h}}\subset\mathfrak{sp}(n)$ is a vector subspace preserving $\mathrm{Im}\mathbb{H}^n$. Let $R\in\mathcal{R}(\mathfrak{g})$ be as in Proposition 1. Then

$$\mathrm{pr}_{\mathbb{H}\oplus\mathfrak{sp}(n)}R(q,I_sq)=C_{0s}+A_{0s}=C_{0s}-\mathrm{Op}(a_{0s}E_n)+B_{0s},$$

where $a_{0s}=\mathrm{Im}C_{0s}$ and $B_{0s}\in\mathfrak{sp}(n)$ preserves $\mathrm{Im}\mathbb{H}^n$. Clearly, B_{0s} preserves $\mathbb{R}^n$. Recall that

$$T_0(X)=-\frac{1}{2}(I_1A_{01}X+I_2A_{02}X+I_3A_{03}X)\in L$$

for any $X \in \mathbb{H}^n$. Let $e_\alpha \in \mathbb{R}^n \subset \mathbb{H}^n$ be an element of the basis. The condition $T_0(e_\alpha) \in L$ implies $\mathrm{Re}(ia_{01} + ja_{02} + ka_{03}) = 0$. The condition $T_0(ie_\alpha) \in L$ implies

$$B_{01}e_\alpha = \mathrm{Re}((ia_{01} + ja_{02} + ka_{03})i)e_\alpha.$$

Since $B_{01} \in \mathfrak{sp}(n) \subset \mathfrak{so}(4n)$, we conclude $\mathrm{Re}((ia_{01}+ja_{02}+ka_{03})i) = 0$ and $B_{01} = 0$. Similarly, $B_{02} = B_{03} = 0$ and

$$\mathrm{Re}((ia_{01} + ja_{02} + ka_{03})j) = \mathrm{Re}((ia_{01} + ja_{02} + ka_{03})k) = 0.$$

Thus, $ia_{01} + ja_{02} + ka_{03} = 0$, i.e. $a_{03} = ja_{01} - ia_{02}$. This and the equalities $a_{0s} = \mathrm{Im}C_{0s}$, $C_{03} = C_{02}i - C_{01}j$ imply $a_{0s} = C_{0s} \in \mathrm{Im}\mathbb{H}$. Thus since $\mathfrak{g}$ is a Berger algebra, $\mathrm{pr}_{\mathbb{R}}\mathfrak{g} = 0$. Lemma 1 shows that either $\bar{\mathfrak{h}} = 0$, or $\bar{\mathfrak{h}} = \{a+\mathrm{Op}(-aE_n)|a \in \mathrm{Im}\mathbb{H}\}$. If we do not assume that $m = 0$, then $\bar{\mathfrak{h}} \subset \{a+\mathrm{Op}(-aE_{m_1})|a \in \mathrm{Im}\mathbb{H}\}\oplus\mathfrak{sp}(m)$ and $\mathrm{pr}_{\mathbb{R}}\mathfrak{g} = 0$, moreover, the projection of $\bar{\mathfrak{h}}$ to $\{a + \mathrm{Op}(-aE_{m_1})|a \in \mathrm{Im}\mathbb{H}\}$ is either trivial or it coincides with $\{a + \mathrm{Op}(-aE_{m_1})|a \in \mathrm{Im}\mathbb{H}\}$.

2.b) Suppose that $m_1 = 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \mathbb{C}^{m_2}$, $m + m_2 = n$. As above, suppose that $m = 0$, then $L = \mathbb{C}^n$. Let $e_1, ..., e_n$ a basis of the complex vector space $L = \mathbb{C}^n$. Let $C + A \in \bar{\mathfrak{h}}$, where $C \in \mathfrak{sp}(1)$ and $A \in \mathfrak{sp}(n)$. Let $A_{\alpha\beta}$ be the matrix of A with respect to the basis $e_1, ..., e_n$ of $\mathbb{H}^n$. Then since $(C + A)e_\alpha \in L$ and $(C + A)ie_\alpha \in L$, we get $C + A_{\alpha\alpha} \in \mathbb{C}$ and $Ci + iA_{\alpha\alpha} \in \mathbb{C}$. Consequently, $C \in \mathbb{C}$. This shows that A preserves L and $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(n)$. Let $R \in \mathcal{R}(\mathfrak{g})$ be as in Proposition 1. Then

$$\mathrm{pr}_{\mathbb{H}\oplus\mathfrak{sp}(n)}R(q, I_s q) = C_{0s} + A_{0s} \in \mathbb{C} \oplus \mathfrak{sp}(n)$$

preserves L and A_{0s} preserves L. By the arguments of the proof of statement 1), $A_{0s} = 0$. Thus, $\bar{\mathfrak{h}} \subset \mathbb{R}i$. If $m \neq 0$, then $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(m)$.

2.c) Suppose that $m_1 \neq 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \mathrm{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2}$, $m + m_1 + m_2 = n$. As in the proof of 2.b), it can be shown that $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(m) \oplus \mathfrak{sp}(m_1)$, i.e. $\mathrm{pr}_{\mathfrak{sp}(1)}\mathfrak{g} \subset \mathbb{R}$. As in the proof of 2.a), it can be proved that $\mathrm{pr}_{\mathbb{R}}\mathfrak{g} = 0$. From this and Lemma 1 it follows that $\mathrm{pr}_{\mathbb{H}}\mathfrak{g} = 0$, i.e. $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m) \oplus \mathfrak{sp}(m_1)$. By the arguments of the proof of 1), $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m)$. The lemma is proved. $\square$

Using the facts summarized above, it is easy to see that all weakly-irreducible Berger subalgebras containing $\mathrm{Im}\,\mathbb{H}$ are exhausted by the Lie algebras given in the statement of the theorem.

3.5. Any weakly-irreducible Berger subalgebra $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ contains $\mathrm{Im}\mathbb{H}$

The following statement was conjectured in [8].

Proposition 2. *Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly-irreducible Berger subalgebra, then it is conjugated to a subalgebra that contains the ideal $\mathcal{I} = \mathrm{Im}\,\mathbb{H}$.*

Proof. We have seen above that if $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ is weakly-irreducible, then $f(\mathfrak{g})$ coincides with one of the Lie algebras $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$. Suppose that $\mathfrak{g}$ is a Berger algebra. The structure of the Lie brackets of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ shows that if $m \neq 0$, or $m_1 \neq 0$, then $\mathfrak{g}$ contains $\mathrm{Im}\,\mathbb{H}$. Thus, $L = \mathbb{C}^{m_2} \oplus L'$. Above we have seen that if $\mathfrak{g}$ is a Berger algebra with such L, then $\mathrm{pr}_{\mathfrak{sp}(n)}\mathfrak{g} = 0$. Hence, if $L' = 0$, then $\bar{\mathfrak{h}} \subset \mathbb{R}i$; if $L' \neq 0$, then $\bar{\mathfrak{h}} = 0$. Recall that since $\mathfrak{g}$ is a Berger algebra, $\dim \mathrm{pr}_{\mathbb{H}}\mathfrak{g} \neq 1$. This shows that either $f(\mathfrak{g}) = \mathfrak{f}_1$ with $\bar{\mathfrak{h}} = \mathbb{R}i$ and $L' = 0$, or $f(\mathfrak{g}) = \mathfrak{f}_2$ with $\varphi = 0$ and $\mathfrak{h} = 0$.

Consider the first case. We have $(1, 0, 0, b) \in \mathfrak{g}$ for some $b \in \mathrm{Im}\,\mathbb{H}$. Let $X \in \mathbb{C}^n$, then $(0, 0, X, c) \in \mathfrak{g}$. Next,

$$[(1, 0, 0, b), (0, 0, X, c)] = (0, 0, X, 2c) \in \mathfrak{g}.$$

This shows that $\mathbb{C}^n \subset \mathfrak{g}$. Consequently,

$$[(0, 0, e_1, 0), (0, 0, ie_1, 0)] = (0, 0, 0, -2i) \in \mathfrak{g}.$$

Hence, $\mathbb{R}(0, 0, 0, i) \subset \mathfrak{g}$. If $(0, 0, 0, \alpha j + \beta k) \in \mathfrak{g}$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$, then taking the Lie bracket of this element with $(i, 0, 0, c) \in \mathfrak{g}$, we get $(0, 0, 0, \alpha k - \beta j) \in \mathfrak{g}$, i.e. $\mathrm{Im}\,\mathbb{H} \subset \mathfrak{g}$. Assume that $\mathfrak{g} \cap \mathbb{R}j \oplus \mathbb{R}k = 0$. Then it is not hard to see that

$$\mathfrak{g} = \mathbb{R}(1, 0, 0, \alpha j + \beta k) \oplus \mathbb{R}(i, 0, 0, -\beta j + \alpha k) \oplus \mathbb{C}^n \ltimes \mathbb{R}(0, 0, 0, i).$$

Let $R \in \mathcal{R}(\mathfrak{g})$. Above we have seen that the elements defining R are zero possibly except for $C_{02} \in \mathbb{C}$, $S_{02} \in \mathbb{C}^n$ and some of D_{rs}. Let $X \in \mathbb{H}^n$. It holds $R(I_s q, X) = (0,0,0,\theta_s(X)) \in \mathfrak{g}$. Hence, $\theta_s(X) \in \mathbb{R}i$. From (21) it follows that $\eta(\theta_s(X)p, I_2 q) = \eta(I_s S_{02}, X)$. Consequently, $S_{02} = 0$, and $\mathfrak{g}$ is not a Berger algebra.

Suppose that $f(\mathfrak{g}) = \mathfrak{f}_2$ and $m_2 \neq 0$. Then for some $b, c \in \operatorname{Im}\mathbb{H}$, $(0,0,e_1,b), (0,0,ie_1,c) \in \mathfrak{g}$. Taking the Lie brackets of these elements, we get $(0,0,0,i) \in \mathfrak{g}$. Using (21), we get that $\eta(\theta_s(X)p, I_2 q) = \eta(I_s S_{02}, X)$. If $\mathfrak{g}$ is a Berger algebra, then for some $R \in \mathcal{R}(\mathfrak{g})$ it holds $S_{02} \neq 0$. There exists $X \in \operatorname{Im}\mathbb{H}$ such that $\eta(S_{02}, X) \neq 0$. Hence, $\theta_0(X) \in \operatorname{Im}\mathbb{H}$ has a non-zero projection to $\mathbb{R}j \subset \operatorname{Im}\mathbb{H}$. Since $R(I_s q, X) = (0,0,0,\theta_s(X)) \in \mathfrak{g}$, there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq 0$ and $(0,0,0,\alpha j + \beta k) \in \mathfrak{g}$. We may assume that $\alpha^2 + \beta^2 = 1$. In [9] it is shown that there exists $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$ and with respect to the new basis with $p' = (x+iy)p$ and $q' = (x+iy)q$ the elements $(0,0,0,i) \in \mathfrak{g}$ and $(0,0,0,\alpha j + \beta k) \in \mathfrak{g}$ have the form $(0,0,0,i)$ and $(0,0,0,j)$, respectively. Note that $S_{03} = -iS_{02} \neq 0$. As above, there exists $X \in \operatorname{Im}\mathbb{H}$ such that $\eta(I_1 S_{03}, X) \neq 0$. Hence, $\eta(\theta_s(X)p, I_3 q) \neq 0$, i.e. $\theta_1(X) \in \operatorname{Im}\mathbb{H}$ has a non-zero projection to $\mathbb{R}k \subset \operatorname{Im}\mathbb{H}$. We conclude that $\mathfrak{g}$ contains $\operatorname{Im}\mathbb{H}$.

Suppose now that $m_2 = 0$, i.e. $L = L'$. Suppose that $\dim \mathfrak{g} \cap \operatorname{Im}\mathbb{H} = 2$. From the results of [9] it follows that choosing an appropriate basis we may get $\mathfrak{g} \cap \operatorname{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j$. Since $R(I_s q, X) = (0,0,0,\theta_s(X))$, we get that $\theta_s(X) \in \mathbb{R}i \oplus \mathbb{R}j$ for any $X \in \mathbb{H}^n$. Since $\theta_s(X) = g(X, S_{0s}) + I_s\theta_0(X)$, we get that for any $X \in \mathbb{H}^n$ and $s = 1,2$ it holds $g(X, S_{0s}) \in \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j$. Since $g(kS_{0s}, S_{0s}) = kg(S_{0s}, S_{0s}) \in \mathbb{R}k$, we get $g(S_{0s}, S_{0s}) = 0$, consequently, $S_{0s} = 0$ for $s = 1,2$. This implies $S_{rs} = 0$. Hence, $\mathfrak{g}$ is not a Berger algebra. The case $\dim \mathfrak{g} \cap \operatorname{Im}\mathbb{H} < 2$ follows from this one. This proves the proposition and the theorem. □ □

4. Pseudo-hyper-Kählerian symmetric spaces of index 4

In [1, 22, 23] indecomposable simply connected pseudo-hyper-Kählerian symmetric spaces of signature $(4, 4n+4)$ are classified. Here we use the results of this paper to give new proof to this result. For $n = 0$ such new proof is obtained in [9].

As it is explained e.g. in [9] the classification of indecomposable simply connected pseudo-hyper-Kählerian symmetric spaces is equivalent to the classification of pairs $(\mathfrak{g}, R)$ (*symmetric pairs*), where $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ is a subalgebra, $R \in \mathcal{R}(\mathfrak{g})$, $R(\mathbb{R}^{4,4n+4}, \mathbb{R}^{4,4n+4}) = \mathfrak{g}$, and for any $\xi \in \mathfrak{g}$ it holds

$$\xi \cdot R = 0, \quad (\xi \cdot R)(x,y) = [\xi, R(x,y)] - R(\xi x, y) - R(x, \xi y), \tag{29}$$

where $x, y \in \mathbb{R}^{4,4n+4}$. *An isomorphism* of symmetric pairs $f : (\mathfrak{g}_1, R_1) \to (\mathfrak{g}_2, R_2)$ consists of an isometry of $\mathbb{R}^{4,4n+4}$ that defines the equivalence of the representations $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{sp}(1, n+1)$ and sends R_1 to R_2. For a positive real number $c \in \mathbb{R}$, the symmetric pairs $(\mathfrak{g}, cR)$ and $(\mathfrak{g}, R)$ define diffeomorphic simply connected symmetric spaces and the metrics of these spaces differ by the factor c. Hence we may identify $(\mathfrak{g}, cR)$ and $(\mathfrak{g}, R)$.

Theorem 2. *Let (M, g) be a non-flat simply connected pseudo-hyper-Kählerian symmetric space of signature $(4, 4n+4)$ $(n \geq 1)$ and $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ its holonomy algebra. Then $n = 2$,*

$$\mathfrak{g} = L' \ltimes \operatorname{Im}\mathbb{H},$$

there exists a basis e_1, e_2 of $\mathbb{H}^2$ such that $L' = \operatorname{span}_{\mathbb{R}}\{e_1, e_2, je_2 + ie_1\}$, the Gram matrix of g with respect to this basis equals to $G = \begin{pmatrix} 1 & -\frac{1}{2}k \\ \frac{1}{2}k & 1 \end{pmatrix}$. The manifold (M, g) is defined by the symmetric pair $(\mathfrak{g}, R)$, where R is defined as in Proposition 1 and it is given by $S_{01} = e_1$, $S_{02} = -e_2$ and other elements defining R are zero.

Proof. Since (M, g) is Ricci-flat, its holonomy algebra $\mathfrak{g}$ cannot be reductive, hence it is conjugated to a subalgebra of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$. Let $(\mathfrak{g}, R)$ be a symmetric pair. Then the tensor R is given as in Proposition 1.

First suppose that $\operatorname{pr}_{\mathbb{H}}\mathfrak{g} \neq 0$. Let $\xi = (a, A, 0, 0) \in \mathfrak{g}$. Then

$$[\xi, R(I_r q, I_s q)] - R(\xi I_r q, I_s q) - R(I_r q, \xi I_s q) = 0.$$

Taking the projection on $\mathbb{H}$, we get the same equations on a, C_{rs} as in [9], where it is shown that these equations imply $C_{rs}=0$. Hence, $R(\mathbb{R}^{4,4n+4},\mathbb{R}^{4,4n+4})\neq\mathfrak{g}$ and we get a contradiction. Thus, $\mathrm{pr}_{\mathbb{H}}\mathfrak{g}=0$.

Since $\mathrm{pr}_{\mathbb{H}}\mathfrak{g}=0$, we get $C_{rs}=B_{rs}=0$. Let $\xi=(0,0,Y,0)\in\mathfrak{g}$ and $X,Z\in\mathbb{H}^n$. The condition $(\xi\cdot R)(X,Z)=0$ implies $R'=0$ and $\operatorname{Im} g(Y,L(X,Z))=0$. Since L is defined by $P_0\in\mathfrak{sp}(m)$, we may take iY instead of Y and get $\operatorname{Im} g(iY,L(X,Z))=0$. This shows that $L=0$ and $P_0=0$. Suppose that $Y\in\mathbb{H}^m$. The condition $(\xi\cdot R)(q,I_sq)=0$ implies

$$-A_{0s}Y+T_s(Y)-T_0(I_sY)=0.$$

Substituting $T_s(Y)=I_sT_0(Y)-A_{0s}(Y)$, we get

$$-2A_{0s}Y+I_sT_0(Y)-T_0(I_sY)=0.$$

Replacing Y by I_sY, multiplying the obtained equality by I_s, and combining it with the last one, we get $A_{0s}Y=0$. Hence, $A_{0s}=0$.

Now R is defined only by S_{rs} and D_{rs} and L must be spanned by S_{01},S_{02},S_{03}. This shows that $\mathfrak{g}=L\ltimes\operatorname{Im}\mathbb{H}$ and L has dimension at most 3. Hence $n=1$ or 2. If $n=1$, then either $L=\operatorname{Im}\mathbb{H}$, or $L=\mathbb{C}^2$. If $n=2$, then $L=L'$ and $\dim L'=3$.

Let $Y\in L$ and $\xi=(0,0,Y,0)\in\mathfrak{g}$. The condition $(\xi\cdot R)(q,I_sq)=0$ implies

$$2\operatorname{Im} g(Y,S_{0s})=\theta_0(I_sY)-\theta_s(Y). \tag{30}$$

Using this, (19) and (20), we get a certain equation on S_{rs}. It can be checked that if $n=1$, then this equation implies $S_{rs}=0$.

Let $n=2$, then $L=L'$ and we get $L'=\{e_1,e_2,je_1+ie_2\}$ for some basis (e_1,e_2) of $\mathbb{H}^2$. Let us find S_{rs}. We may write

$$S_{0s}=a_se_1+b_se_2+c_s(je_1+ie_2),$$

where $a_s,b_s,c_s\in\mathbb{R}$. The condition $S_{03}=jS_{01}-iS_{02}$ implies $S_{01}=a_1e_1$ and $S_{02}=-a_1e_2$. We may assume that $a_1=1$. Using this, (30), (19) and (20), it is not hard to get $G=\begin{pmatrix} a & -\frac{1}{2}ak\\ \frac{1}{2}ak & a\end{pmatrix}$ for some $a\in\mathbb{R}$, $a>0$. Changing e_1,e_2 by $\frac{\sqrt{a}}{a}e_1,\frac{\sqrt{a}}{a}e_2$, we get $G=\begin{pmatrix} 1 & -\frac{1}{2}k\\ \frac{1}{2}k & 1\end{pmatrix}$.

Note that we still have possibly non-zero elements D_{rs}. Let us consider $q'=-\frac{1}{2}p+X+q$ for some $X\in\mathbb{H}^n$. This defines a new basis of $\mathbb{H}^{1,3}$. With respect to this basis R is given by the elements S'_{rs} and D'_{rs}. Note that

$$D'_{rs}=D_{rs}-\theta_s(I_rX)+\theta_r(I_sX)-g(X,S_{rs})+g(S_{rs},X).$$

It can be shown that there exists $X\in\mathbb{H}^n$ such that $D'_{01}=D'_{02}=0$. Then, $D'_{rs}=0$. Beginning with such basis, the above arguments prove the theorem. □

It can be checked that the symmetric space from the above theorem coincides with the one obtained in [22, 23].

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